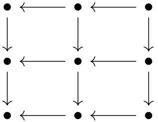
Notation

Throughout:

- Chain complexes are $C_\bullet = C$, or $C_{\bullet,\bullet}$.
- Homology of a complex C is $h_n(C)$.
- Double complexes are almost always homologically oriented:

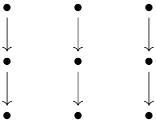


$d^<$ has bidegree $(-1, 0)$ and $d^\vee$ has bidegree $(0, -1)$.

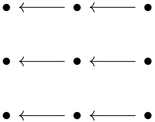
and almost always first quadrant.

- EC^r_{pq} is the (homological) vertical filtration of a double complex and ER^r_{pq} is the horizontal filtration.

EC^0_{pq}:



ER^0_{qp}:



- We will transpose horizontal filtrations to orient like vertical filtrations so that we need not worry about the interchange of indices.
- Vertical and horizontal homology of a double complex C are denoted $H^v(C)$ and $H^h(C)$, respectively.

Recall

A spectral sequence is a collection of modules $\{E^r_{pq}\}$ for all $p, q \in \mathbf{Z}$ (for us $p \geq 0$ often and $q \geq 0$ always) and $r \geq a$ (for us $a = 0$) such that

- For each r there exist differentials $d^r : E^r_{pq} \to E^r_{p-r,q+r-1}$. That is, arrows on page r go r left and $r-1$ up.
- There are isomorphisms $E^{r+1}_{pq} \cong \ker d^r_{pq} / \operatorname{im} d^r_{p+r,q-r+1}$. That is, objects on page $r+1$ are homology modules of the objects on page r.

A double complex C that is first quadrant has bounded filtrations (both vertical and horizontal), and thus by last time

$$EC^2_{pq} \Rightarrow h_{p+q}(\mathrm{Tot}^{\oplus}(C)) \Leftarrow ER^2_{pq}.$$

Outline

Balancing Tor

The left-derived functors $\mathbf{L}_n(A \otimes_R -)(B)$ and $\mathbf{L}_n(- \otimes_R B)(A)$ are isomorphic; we call both $\mathrm{Tor}_n^R(A, B)$.

Balancing Tor

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Universal Coefficient Theorem

If C is a complex of free abelian groups and A is an abelian group, then there exists a split short exact sequence

$$0 \to h_n(C) \otimes A \to h_n(C \otimes A) \to \mathrm{Tor}_1(h_{n-1}(C), A) \to 0.$$

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Balancing Ext

The right-derived functors $\mathbf{R}^n \operatorname{Hom}_R(A, -)(B)$ and $\mathbf{R}^n \operatorname{Hom}_R(-, B)(A)$ are isomorphic; we call both $\operatorname{Ext}_R^n(A, B)$.

Balancing Tor

Let A and B be R-modules, so A has projective resolution

$$\cdots \xrightarrow{d_P} P_3 \xrightarrow{d_P} P_2 \xrightarrow{d_P} P_1 \xrightarrow{d_P} P_0 \xrightarrow{\varepsilon_A} A \to 0$$

and B has projective resolution

$$\cdots \xrightarrow{d_Q} Q_3 \xrightarrow{d_Q} Q_2 \xrightarrow{d_Q} Q_1 \xrightarrow{d_Q} Q_0 \xrightarrow{\varepsilon_B} B \to 0.$$

To compute the left-derived functor $\mathbf{L}_n(A \otimes -)(B)$, we can compute (independent of choice of $Q_\bullet$)

$$\mathbf{L}_n(A \otimes -)(B) = h_n(A \otimes Q_\bullet)$$

and similarly,

$$\mathbf{L}_n(- \otimes B)(A) = h_n(P_\bullet \otimes B).$$

Balancing Tor

Goal: show that

$$\mathbf{L}_n(A \otimes -)(B) \cong \mathbf{L}_n(- \otimes B)(A)$$

for all n.

Balancing Tor

Step One: Build the double complex $P \otimes Q$.

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & P_0 \otimes Q_3 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_1 \otimes Q_3 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_2 \otimes Q_3 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_3 \otimes Q_3 & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & & \\
0 & \longleftarrow & P_0 \otimes Q_2 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_1 \otimes Q_2 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_2 \otimes Q_2 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_3 \otimes Q_2 & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & & \\
0 & \longleftarrow & P_0 \otimes Q_1 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_1 \otimes Q_1 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_2 \otimes Q_1 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_3 \otimes Q_1 & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & & \\
0 & \longleftarrow & P_0 \otimes Q_0 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_1 \otimes Q_0 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_2 \otimes Q_0 & \xleftarrow[d_P \otimes \mathrm{id}]{} & P_3 \otimes Q_0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Step Two: Take the vertical filtration EC^0_{pq}.

$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & P_0 \otimes Q_3 & P_1 \otimes Q_3 & P_2 \otimes Q_3 & P_3 \otimes Q_3 & \cdots \\
 & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & \\
0 & P_0 \otimes Q_2 & P_1 \otimes Q_2 & P_2 \otimes Q_2 & P_3 \otimes Q_2 & \cdots \\
 & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & \\
0 & P_0 \otimes Q_1 & P_1 \otimes Q_1 & P_2 \otimes Q_1 & P_3 \otimes Q_1 & \cdots \\
 & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes d_Q} & \downarrow{\scriptstyle \mathrm{id} \otimes -d_Q} & \\
0 & P_0 \otimes Q_0 & P_1 \otimes Q_0 & P_2 \otimes Q_0 & P_3 \otimes Q_0 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & 0 & 0 & 0 & 0 &
\end{array}
$$

Step Two: Take the vertical filtration EC^0_{pq}.

$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & P_0 \otimes Q_3 & P_1 \otimes Q_3 & P_2 \otimes Q_3 & P_3 \otimes Q_3 & \cdots \\
 & \downarrow d_Q & \downarrow d_Q & \downarrow d_Q & \downarrow d_Q & \\
0 & P_0 \otimes Q_2 & P_1 \otimes Q_2 & P_2 \otimes Q_2 & P_3 \otimes Q_2 & \cdots \\
 & \downarrow d_Q & \downarrow d_Q & \downarrow d_Q & \downarrow d_Q & \\
0 & P_0 \otimes Q_1 & P_1 \otimes Q_1 & P_2 \otimes Q_1 & P_3 \otimes Q_1 & \cdots \\
 & \downarrow d_Q & \downarrow d_Q & \downarrow d_Q & \downarrow d_Q & \\
0 & P_0 \otimes Q_0 & P_1 \otimes Q_0 & P_2 \otimes Q_0 & P_3 \otimes Q_0 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & 0 & 0 & 0 & 0 &
\end{array}
$$

Balancing Tor

Since each P_i is projective, the following sequence is exact for all i:

$$\begin{array}{c} \vdots \\ \downarrow \\ P_i \otimes Q_3 \\ {\scriptstyle d_Q}\downarrow \\ P_i \otimes Q_2 \\ {\scriptstyle d_Q}\downarrow \\ P_i \otimes Q_1 \\ {\scriptstyle d_Q}\downarrow \\ P_i \otimes Q_0 \\ \downarrow \\ P_i \otimes B \\ \downarrow \\ 0 \end{array}$$

Hence $H^v(P \otimes Q) = h_n(P_i \otimes Q) = 0$ for all $n \neq 0$, and

$$h_0(P_i \otimes Q) = \operatorname{coker}\left(P_i \otimes Q_1 \xrightarrow{d_Q} P_i \otimes Q_0\right) = P_i \otimes B.$$

Balancing Tor

Step Three: Build page 1, where $EC^1_{pq} = H^v(P \otimes Q)$, and d^1 goes 1 left and $1 - 1 = 0$ up.

$$
\begin{array}{cccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
0 & \longleftarrow & P_0 \otimes B & \longleftarrow & P_1 \otimes B & \longleftarrow & P_2 \otimes B & \longleftarrow & P_3 \otimes B & \longleftarrow & \cdots \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Balancing Tor

$$0 \longleftarrow P_0 \otimes B \longleftarrow P_1 \otimes B \longleftarrow P_2 \otimes B \longleftarrow P_3 \otimes B \longleftarrow \cdots$$

Computing $h_n(P \otimes B)$ is, by definition, $\mathbf{L}_n(- \otimes B)(A)$, since P is a projective resolution of A. Hence our page 2 looks like

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \\ 0 & \mathbf{L}_0(- \otimes B)(A) & \mathbf{L}_1(- \otimes B)(A) & \mathbf{L}_2(- \otimes B)(A) & \mathbf{L}_3(- \otimes B)(A) & \cdots \\ 0 & 0 & 0 & 0 & 0 & \end{array}$$

At this point, our homology has stabilized, since for any n, page 2 has

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ & & \mathbf{L}_n(- \otimes B)(A) & & \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

and computing homology just returns $\mathbf{L}_n(- \otimes B)(A)$. All subsequent pages are as above, and

$$EC^2_{pq} = H^h_p(H^v_q(P \otimes Q)) \Rightarrow h_{p+q}(\mathrm{Tot}^{\oplus}(P \otimes Q)).$$

Balancing Tor

Step Four: Take the horizontal filtration ER^0_{pq}.

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \longleftarrow & P_0 \otimes Q_3 & \xleftarrow{d_P \otimes \mathrm{id}} & P_1 \otimes Q_3 & \xleftarrow{d_P \otimes \mathrm{id}} & P_2 \otimes Q_3 & \xleftarrow{d_P \otimes \mathrm{id}} & P_3 \otimes Q_3 & \longleftarrow & \cdots \\
0 & \longleftarrow & P_0 \otimes Q_2 & \xleftarrow{d_P \otimes \mathrm{id}} & P_1 \otimes Q_2 & \xleftarrow{d_P \otimes \mathrm{id}} & P_2 \otimes Q_2 & \xleftarrow{d_P \otimes \mathrm{id}} & P_3 \otimes Q_2 & \longleftarrow & \cdots \\
0 & \longleftarrow & P_0 \otimes Q_1 & \xleftarrow{d_P \otimes \mathrm{id}} & P_1 \otimes Q_1 & \xleftarrow{d_P \otimes \mathrm{id}} & P_2 \otimes Q_1 & \xleftarrow{d_P \otimes \mathrm{id}} & P_3 \otimes Q_1 & \longleftarrow & \cdots \\
0 & \longleftarrow & P_0 \otimes Q_0 & \xleftarrow{d_P \otimes \mathrm{id}} & P_1 \otimes Q_0 & \xleftarrow{d_P \otimes \mathrm{id}} & P_2 \otimes Q_0 & \xleftarrow{d_P \otimes \mathrm{id}} & P_3 \otimes Q_0 & \longleftarrow & \cdots \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Balancing Tor

Step Four B: Reorient to avoid fiddling with indices.

$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & P_3 \otimes Q_0 & P_3 \otimes Q_1 & P_3 \otimes Q_2 & P_3 \otimes Q_3 & \cdots \\
 & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \\
0 & P_2 \otimes Q_0 & P_2 \otimes Q_1 & P_2 \otimes Q_2 & P_2 \otimes Q_3 & \cdots \\
 & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \\
0 & P_1 \otimes Q_0 & P_1 \otimes Q_1 & P_1 \otimes Q_2 & P_1 \otimes Q_3 & \cdots \\
 & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \downarrow{\scriptstyle d_P \otimes \mathrm{id}} & \\
0 & P_0 \otimes Q_0 & P_0 \otimes Q_1 & P_0 \otimes Q_2 & P_0 \otimes Q_3 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & 0 & 0 & 0 & 0 &
\end{array}
$$

Step Four B: Reorient to avoid fiddling with indices.

$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & P_3 \otimes Q_0 & P_3 \otimes Q_1 & P_3 \otimes Q_2 & P_3 \otimes Q_3 & \cdots \\
 & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \\
0 & P_2 \otimes Q_0 & P_2 \otimes Q_1 & P_2 \otimes Q_2 & P_2 \otimes Q_3 & \cdots \\
 & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \\
0 & P_1 \otimes Q_0 & P_1 \otimes Q_1 & P_1 \otimes Q_2 & P_1 \otimes Q_3 & \cdots \\
 & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \downarrow{\scriptstyle d_P} & \\
0 & P_0 \otimes Q_0 & P_0 \otimes Q_1 & P_0 \otimes Q_2 & P_0 \otimes Q_3 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & 0 & 0 & 0 & 0 &
\end{array}
$$

Balancing Tor

Since each Q_i is projective, the following sequence is exact for all i:

$$\begin{array}{c} \vdots \\ \downarrow \\ P_3 \otimes Q_i \\ \downarrow d_P \\ P_2 \otimes Q_i \\ \downarrow d_P \\ P_1 \otimes Q_i \\ \downarrow d_P \\ P_0 \otimes Q_i \\ \downarrow \\ A \otimes Q_i \\ \downarrow \\ 0 \end{array}$$

Hence $H^h(P \otimes Q) = h_n(P \otimes Q_i) = 0$ for all $n \neq 0$, and

$$h_0(P \otimes Q_i) = \operatorname{coker}\left(P_1 \otimes Q_i \xrightarrow{d_P} P_0 \otimes Q_i\right) = A \otimes Q_i.$$

Balancing Tor

Step Five: Build page 1, where $ER^1_{pq} = H^h(P \otimes Q)$, and d^1 goes 1 left and $1 - 1 = 0$ up.

$$
\begin{array}{cccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
0 & \longleftarrow & A \otimes Q_0 & \longleftarrow & A \otimes Q_1 & \longleftarrow & A \otimes Q_2 & \longleftarrow & A \otimes Q_3 & \longleftarrow & \cdots \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Balancing Tor

$$0 \longleftarrow A \otimes Q_0 \longleftarrow A \otimes Q_1 \longleftarrow A \otimes Q_2 \longleftarrow A \otimes Q_3 \longleftarrow \cdots$$

Computing $h_n(A \otimes Q)$ is, by definition, $\mathbf{L}_n(A \otimes -)(B)$, since Q is a projective resolution of B. Hence our page 2 looks like

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \\ 0 & \mathbf{L}_0(A \otimes -)(B) & \mathbf{L}_1(A \otimes -)(B) & \mathbf{L}_2(A \otimes -)(B) & \mathbf{L}_3(A \otimes -)(B) & \cdots \\ 0 & 0 & 0 & 0 & 0 & \end{array}$$

At this point, our homology has stabilized, since for any n, page 2 has

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ & & \mathbf{L}_n(A \otimes -)(B) & & \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

and computing homology just returns $\mathbf{L}_n(A \otimes -)(B)$. All subsequent pages are as above, and

$$ER^2_{pq} = H^v_p(H^h_q(P \otimes Q)) \Rightarrow h_{p+q}(\mathrm{Tot}^{\oplus}(P \otimes Q)).$$

Balancing Tor

Our reorientation did not disturb the total degree of any entry, since we only interchanged p and q. Hence since both

$$EC^2_{pq} \Rightarrow h_{p+q}(\mathrm{Tot}^{\oplus}(P \otimes Q)) \Leftarrow ER^2_{pq}$$

and both spectral sequences collapse having only $\mathbf{L}_n(- \otimes B)(A)$ or $\mathbf{L}_n(A \otimes -)(B)$ in total degree n, we see that

$$\mathbf{L}_n(A \otimes -)(B) \cong h_n(\mathrm{Tot}^{\oplus}(P \otimes Q)) \cong \mathbf{L}_n(- \otimes B)(A),$$

as desired.

Universal Coefficient Theorem

Given an abelian group A and a complex C:

$$\cdots \xrightarrow{d_C} C_{n+1} \xrightarrow{d_C} C_n \xrightarrow{d_C} C_{n-1} \xrightarrow{d_C} \cdots ,$$

take a projective resolution of A:

$$\cdots \xrightarrow{d_P} P_3 \xrightarrow{d_P} P_2 \xrightarrow{d_P} P_1 \xrightarrow{d_P} P_0 \xrightarrow{\varepsilon_A} A \to 0.$$

Universal Coefficient Theorem

We wish to compute $h_n(C_\bullet \otimes A)$. The universal coefficient theorem will relate this homology to the homology of the chain complex $h_n(C)$. By definition,

$$h_n(C) = {}^{\ker d_C}\!\big/_{\operatorname{im} d_C}.$$

Thus, we have a short exact sequence

$$0 \to \operatorname{im} d_C \to \ker d_C \to h_n(C) \to 0.$$

Universal Coefficient Theorem

$$0 \to \operatorname{im} d_C \to \ker d_C \to h_n(C) \to 0$$

Since C is assumed to be comprised of free abelian groups and subgroups of free abelian groups are free abelian, $\ker d_C$ and $\operatorname{im} d_C$ are free abelian. Hence the above short exact sequence is a free (projective, flat) resolution of $h_n(C)$, and thus

$$\mathrm{Tor}_{i\geq 2}(h_n(C), A) = 0$$
$$\mathrm{Tor}_1(h_n(C), A)$$
$$\mathrm{Tor}_0(h_n(C), A) \cong h_n(C) \otimes A$$

Universal Coefficient Theorem

Once again, build a tensor double complex $C \otimes P$.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{n-1} \otimes P_3 & \longleftarrow & C_n \otimes P_3 & \longleftarrow & C_{n+1} \otimes P_3 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{n-1} \otimes P_2 & \longleftarrow & C_n \otimes P_2 & \longleftarrow & C_{n+1} \otimes P_2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{n-1} \otimes P_1 & \longleftarrow & C_n \otimes P_1 & \longleftarrow & C_{n+1} \otimes P_1 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{n-1} \otimes P_0 & \longleftarrow & C_n \otimes P_0 & \longleftarrow & C_{n+1} \otimes P_0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Universal Coefficient Theorem

Let's build ER^0_{pq}.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longleftarrow & C_{n-1} \otimes P_3 & \longleftarrow & C_n \otimes P_3 & \longleftarrow & C_{n+1} \otimes P_3 & \longleftarrow & \cdots \\
\cdots & \longleftarrow & C_{n-1} \otimes P_2 & \longleftarrow & C_n \otimes P_2 & \longleftarrow & C_{n+1} \otimes P_2 & \longleftarrow & \cdots \\
\cdots & \longleftarrow & C_{n-1} \otimes P_1 & \longleftarrow & C_n \otimes P_1 & \longleftarrow & C_{n+1} \otimes P_1 & \longleftarrow & \cdots \\
\cdots & \longleftarrow & C_{n-1} \otimes P_0 & \longleftarrow & C_n \otimes P_0 & \longleftarrow & C_{n+1} \otimes P_0 & \longleftarrow & \cdots \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Universal Coefficient Theorem

And again transpose.

$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & C_{n+1} \otimes P_0 & C_{n+1} \otimes P_1 & C_{n+1} \otimes P_2 & C_{n+1} \otimes P_3 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & C_n \otimes P_0 & C_n \otimes P_1 & C_n \otimes P_2 & C_n \otimes P_3 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & C_{n-1} \otimes P_0 & C_{n-1} \otimes P_1 & C_{n-1} \otimes P_2 & C_{n-1} \otimes P_3 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Universal Coefficient Theorem

And again transpose.

$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & C_{n+1} \otimes P_0 & C_{n+1} \otimes P_1 & C_{n+1} \otimes P_2 & C_{n+1} \otimes P_3 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & C_n \otimes P_0 & C_n \otimes P_1 & C_n \otimes P_2 & C_n \otimes P_3 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & C_{n-1} \otimes P_0 & C_{n-1} \otimes P_1 & C_{n-1} \otimes P_2 & C_{n-1} \otimes P_3 & \cdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

As P_i is projective, $H^h(C \otimes P) = h_n(C \otimes P_i) = h_n(C) \otimes P_i$.

Universal Coefficient Theorem

Hence we have ER^1_{pq}:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longleftarrow & h_{n+1}(C)\otimes P_0 & \longleftarrow & h_{n+1}(C)\otimes P_1 & \longleftarrow & h_{n+1}(C)\otimes P_2 & \longleftarrow & h_{n+1}(C)\otimes P_3 & \longleftarrow & \cdots \\
0 & \longleftarrow & h_{n}(C)\otimes P_0 & \longleftarrow & h_{n}(C)\otimes P_1 & \longleftarrow & h_{n}(C)\otimes P_2 & \longleftarrow & h_{n}(C)\otimes P_3 & \longleftarrow & \cdots \\
0 & \longleftarrow & h_{n-1}(C)\otimes P_0 & \longleftarrow & h_{n-1}(C)\otimes P_1 & \longleftarrow & h_{n-1}(C)\otimes P_2 & \longleftarrow & h_{n-1}(C)\otimes P_3 & \longleftarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Universal Coefficient Theorem

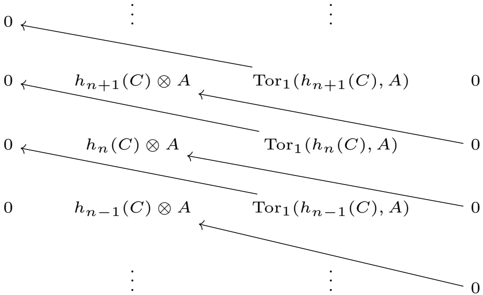
Hence we have ER^1_{pq}:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \longleftarrow & h_{n+1}(C)\otimes P_0 & \longleftarrow & h_{n+1}(C)\otimes P_1 & \longleftarrow & h_{n+1}(C)\otimes P_2 & \longleftarrow & h_{n+1}(C)\otimes P_3 & \longleftarrow & \cdots \\
0 & \longleftarrow & h_{n}(C)\otimes P_0 & \longleftarrow & h_{n}(C)\otimes P_1 & \longleftarrow & h_{n}(C)\otimes P_2 & \longleftarrow & h_{n}(C)\otimes P_3 & \longleftarrow & \cdots \\
0 & \longleftarrow & h_{n-1}(C)\otimes P_0 & \longleftarrow & h_{n-1}(C)\otimes P_1 & \longleftarrow & h_{n-1}(C)\otimes P_2 & \longleftarrow & h_{n-1}(C)\otimes P_3 & \longleftarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

By definition, the homology of an above row is $\mathrm{Tor}_i(h_n(C), A)$. By prior work, we know what $\mathrm{Tor}_i(h_n(C), A)$ is. Hence we can write page 2:

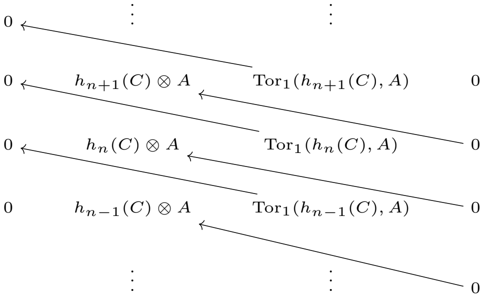
Universal Coefficient Theorem

ER^2_{pq}:



Universal Coefficient Theorem

ER^2_{pq}:



Notice that on page 2 and all subsequent pages, the homology stabilizes, since page 2 is supported in two columns and all differentials will move $r \geq 2$ left. Hence we are taking homology of

$$0 \to M \to 0,$$

which just gives M again.

Universal Coefficient Theorem

Therefore ER^{∞}_{pq}:

$$\begin{array}{cccc}
 & \vdots & \vdots & \\
0 & h_{n+1}(C) \otimes A & \mathrm{Tor}_1(h_{n+1}(C), A) & 0 \\
0 & h_n(C) \otimes A & \mathrm{Tor}_1(h_n(C), A) & 0 \\
0 & h_{n-1}(C) \otimes A & \mathrm{Tor}_1(h_{n-1}(C), A) & 0 \\
 & \vdots & \vdots &
\end{array}$$

Universal Coefficient Theorem

Therefore ER^{∞}_{pq}:

$$
\begin{array}{cccc}
 & \vdots & \vdots & \\
0 & h_{n+1}(C) \otimes A & \mathrm{Tor}_1(h_{n+1}(C), A) & 0 \\
0 & h_n(C) \otimes A & \mathrm{Tor}_1(h_n(C), A) & 0 \\
0 & h_{n-1}(C) \otimes A & \mathrm{Tor}_1(h_{n-1}(C), A) & 0 \\
 & \vdots & \vdots &
\end{array}
$$

Since page infinity also gives us homology of the totalization which is $h_n(C \otimes A)$, we see that

Universal Coefficient Theorem

Therefore ER^{∞}_{pq}:

$$
\begin{array}{cccc}
 & \vdots & \vdots & \\
0 & h_{n+1}(C)\otimes A & \mathrm{Tor}_1(h_{n+1}(C),A) & 0 \\
0 & h_n(C)\otimes A & \mathrm{Tor}_1(h_n(C),A) & 0 \\
0 & h_{n-1}(C)\otimes A & \mathrm{Tor}_1(h_{n-1}(C),A) & 0 \\
 & \vdots & \vdots &
\end{array}
$$

Since page infinity also gives us homology of the totalization which is $h_n(C \otimes A)$, we see that

$$h_n(C \otimes A) \cong h_n(C) \otimes A \oplus \mathrm{Tor}_1(h_{n-1}(C), A),$$

as desired.

Balancing Ext

Let A and B be R-modules, so A has projective resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to A \to 0$$

and B has injective resolution

$$0 \to B \to I^0 \to I^1 \to I^2 \to I^3 \to \cdots$$

To compute the right-derived (covariant) functor $\mathbf{R}^n \operatorname{Hom}(A, -)(B)$, we can compute

$$\mathbf{R}^n \operatorname{Hom}(A, -)(B) = h^n(\operatorname{Hom}(A, I^\bullet))$$

and to compute the right-derived (contravariant) functor $\mathbf{R}^n \operatorname{Hom}(-, B)(A)$, we can compute

$$\mathbf{R}^n \operatorname{Hom}(-, B)(A) = h^n(\operatorname{Hom}(P_\bullet, B)).$$

Balancing Ext

Recall that

$$\mathbf{R}^{i\geq 1}\operatorname{Hom}(A,-)(B) = 0$$

if B is injective and

$$\mathbf{R}^{i\geq 1}\operatorname{Hom}(-,B)(A) = 0$$

if A is projective, since

$$0 \to B \to B \to 0$$

and

$$0 \to A \to A \to 0$$

are injective/projective resolutions.

Balancing Ext

Just like with $\otimes$, we can build a Hom double complex. We build $\mathrm{Hom}(P, I)$ (differentials suppressed). Note it is cohomologically indexed.

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0, I^3) & \longrightarrow & \mathrm{Hom}(P_1, I^3) & \longrightarrow & \mathrm{Hom}(P_2, I^3) & \longrightarrow & \mathrm{Hom}(P_3, I^3) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0, I^2) & \longrightarrow & \mathrm{Hom}(P_1, I^2) & \longrightarrow & \mathrm{Hom}(P_2, I^2) & \longrightarrow & \mathrm{Hom}(P_3, I^2) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0, I^1) & \longrightarrow & \mathrm{Hom}(P_1, I^1) & \longrightarrow & \mathrm{Hom}(P_2, I^1) & \longrightarrow & \mathrm{Hom}(P_3, I^1) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0, I^0) & \longrightarrow & \mathrm{Hom}(P_1, I^0) & \longrightarrow & \mathrm{Hom}(P_2, I^0) & \longrightarrow & \mathrm{Hom}(P_3, I^0) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Balancing Ext

Take a vertical filtration EC_0^{pq}. (Arrows in cohomological spectral sequences on page r will go r right and $r-1$ down.)

$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
0 & \mathrm{Hom}(P_0, I^3) & \mathrm{Hom}(P_1, I^3) & \mathrm{Hom}(P_2, I^3) & \mathrm{Hom}(P_3, I^3) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
0 & \mathrm{Hom}(P_0, I^2) & \mathrm{Hom}(P_1, I^2) & \mathrm{Hom}(P_2, I^2) & \mathrm{Hom}(P_3, I^2) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
0 & \mathrm{Hom}(P_0, I^1) & \mathrm{Hom}(P_1, I^1) & \mathrm{Hom}(P_2, I^1) & \mathrm{Hom}(P_3, I^1) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
0 & \mathrm{Hom}(P_0, I^0) & \mathrm{Hom}(P_1, I^0) & \mathrm{Hom}(P_2, I^0) & \mathrm{Hom}(P_3, I^0) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
 & 0 & 0 & 0 & 0 &
\end{array}
$$

Balancing Ext

For any i, we see that

$$\begin{array}{c} \vdots \\ \uparrow \\ \mathrm{Hom}(P_i, I^3) \\ \uparrow \\ \mathrm{Hom}(P_i, I^2) \\ \uparrow \\ \mathrm{Hom}(P_i, I^1) \\ \uparrow \\ \mathrm{Hom}(P_i, I^0) \\ \uparrow \\ 0 \end{array}$$

$H_v(\mathrm{Hom}(P, I)) = h^n(\mathrm{Hom}(P_i, I)) = 0$ for $n \neq 0$ since P_i is projective, and $h^0(\mathrm{Hom}(P_i, I)) = \mathrm{Hom}(P_i, B)$. Hence we can write page 1:

Balancing Ext

Page 1, EC_1^{pq}:

$$\begin{array}{ccccccccccc} 0 & & 0 & & 0 & & 0 & & 0 & & \\ 0 & \longrightarrow & \operatorname{Hom}(P_0, B) & \longrightarrow & \operatorname{Hom}(P_1, B) & \longrightarrow & \operatorname{Hom}(P_2, B) & \longrightarrow & \operatorname{Hom}(P_3, B) & \longrightarrow & \cdots \\ 0 & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

Balancing Ext

Page 1, EC_1^{pq}:

$$
\begin{array}{cccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & \\
0 & \longrightarrow & \operatorname{Hom}(P_0, B) & \longrightarrow & \operatorname{Hom}(P_1, B) & \longrightarrow & \operatorname{Hom}(P_2, B) & \longrightarrow & \operatorname{Hom}(P_3, B) & \longrightarrow \cdots \\
0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

Take cohomology here to get page 2, which we see will henceforth stabilize as $EC_2^{pq} = EC_3^{pq} = \cdots = EC_\infty^{pq}$:

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{R}^0 \operatorname{Hom}(-, B)(A) & \mathbf{R}^1 \operatorname{Hom}(-, B)(A) & \mathbf{R}^2 \operatorname{Hom}(-, B)(A) & \mathbf{R}^3 \operatorname{Hom}(-, B)(A) & \cdots \\
0 & 0 & 0 & 0 & 0 &
\end{array}
$$

since $h^n(\operatorname{Hom}(P_\bullet, B)) = \mathbf{R}^n \operatorname{Hom}(-, B)(A)$.

On the other hand, ER_0^{pq}:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0, I^3) & \longrightarrow & \mathrm{Hom}(P_1, I^3) & \longrightarrow & \mathrm{Hom}(P_2, I^3) & \longrightarrow & \mathrm{Hom}(P_3, I^3) & \longrightarrow & \cdots \\
0 & \longrightarrow & \mathrm{Hom}(P_0, I^2) & \longrightarrow & \mathrm{Hom}(P_1, I^2) & \longrightarrow & \mathrm{Hom}(P_2, I^2) & \longrightarrow & \mathrm{Hom}(P_3, I^2) & \longrightarrow & \cdots \\
0 & \longrightarrow & \mathrm{Hom}(P_0, I^1) & \longrightarrow & \mathrm{Hom}(P_1, I^1) & \longrightarrow & \mathrm{Hom}(P_2, I^1) & \longrightarrow & \mathrm{Hom}(P_3, I^1) & \longrightarrow & \cdots \\
0 & \longrightarrow & \mathrm{Hom}(P_0, I^0) & \longrightarrow & \mathrm{Hom}(P_1, I^0) & \longrightarrow & \mathrm{Hom}(P_2, I^0) & \longrightarrow & \mathrm{Hom}(P_3, I^0) & \longrightarrow & \cdots \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Balancing Ext

On the other hand, ER_0^{pq} (reoriented):

$$
\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
0 & \mathrm{Hom}(P_3, I^0) & \mathrm{Hom}(P_3, I^1) & \mathrm{Hom}(P_3, I^2) & \mathrm{Hom}(P_3, I^3) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
0 & \mathrm{Hom}(P_2, I^0) & \mathrm{Hom}(P_2, I^1) & \mathrm{Hom}(P_2, I^2) & \mathrm{Hom}(P_2, I^3) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
0 & \mathrm{Hom}(P_1, I^0) & \mathrm{Hom}(P_1, I^1) & \mathrm{Hom}(P_1, I^2) & \mathrm{Hom}(P_1, I^3) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
0 & \mathrm{Hom}(P_0, I^0) & \mathrm{Hom}(P_0, I^1) & \mathrm{Hom}(P_0, I^2) & \mathrm{Hom}(P_0, I^3) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \\
 & 0 & 0 & 0 & 0 &
\end{array}
$$

Balancing Ext

For any i, we see that

$$\begin{array}{c} \vdots \\ \uparrow \\ \mathrm{Hom}(P_3, I^i) \\ \uparrow \\ \mathrm{Hom}(P_2, I^i) \\ \uparrow \\ \mathrm{Hom}(P_1, I^i) \\ \uparrow \\ \mathrm{Hom}(P_0, I^i) \\ \uparrow \\ 0 \end{array}$$

$H_h(\mathrm{Hom}(P, I)) = h^n(\mathrm{Hom}(P, I^i)) = 0$ for $n \neq 0$ since I^i is injective, and $h^0(\mathrm{Hom}(P, I^i)) = \mathrm{Hom}(A, I^i)$. Hence we can write page 1:

Balancing Ext

Page 1, ER_1^{pq}:

$$
\begin{array}{ccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & \\
0 & \longrightarrow & \mathrm{Hom}(A, I^0) & \longrightarrow & \mathrm{Hom}(A, I^1) & \longrightarrow & \mathrm{Hom}(A, I^2) & \longrightarrow & \mathrm{Hom}(A, I^3) & \longrightarrow & \cdots \\
0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Balancing Ext

Page 1, ER_1^{pq}:

$$\begin{array}{cccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & \\
0 & \longrightarrow & \mathrm{Hom}(A, I^0) & \longrightarrow & \mathrm{Hom}(A, I^1) & \longrightarrow & \mathrm{Hom}(A, I^2) & \longrightarrow & \mathrm{Hom}(A, I^3) & \longrightarrow \cdots \\
0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

Take cohomology here to get page 2, which we see will henceforth stabilize as $ER_2^{pq} = ER_3^{pq} = \cdots = ER_\infty^{pq}$:

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{R}^0 \mathrm{Hom}(A, -)(B) & \mathbf{R}^1 \mathrm{Hom}(A, -)(B) & \mathbf{R}^2 \mathrm{Hom}(A, -)(B) & \mathbf{R}^3 \mathrm{Hom}(A, -)(B) & \cdots \\
0 & 0 & 0 & 0 & 0 &
\end{array}$$

since $h^n(\mathrm{Hom}(A, I^\bullet)) = \mathbf{R}^n \mathrm{Hom}(A, -)(B)$.

Balancing Ext

Therefore, we have that since

$$EC \Rightarrow h^n(\mathrm{Tot}^{\oplus}(\mathrm{Hom}(P, I))) \Leftarrow ER,$$

and both EC and ER collapse, we get in degree n

$$\mathbf{R}^n \mathrm{Hom}(-, B)(A) \cong h^n(\mathrm{Tot}^{\oplus}(\mathrm{Hom}(P, I))) \cong \mathbf{R}^n \mathrm{Hom}(A, -)(B).$$

Balancing Ext

Therefore, we have that since

$$EC \Rightarrow h^n(\mathrm{Tot}^{\oplus}(\mathrm{Hom}(P, I))) \Leftarrow ER,$$

and both EC and ER collapse, we get in degree n

$$\mathbf{R}^n \operatorname{Hom}(-, B)(A) \cong h^n(\mathrm{Tot}^{\oplus}(\mathrm{Hom}(P, I))) \cong \mathbf{R}^n \operatorname{Hom}(A, -)(B).$$

"Perfectly balanced, as all things should be."